Model · Answer (AV-8878)

B.Sc. (H.) V^{th} Sem. Exam. 2015-16.

Mathematics

Linear Algebra

1. (i) Let $\alpha \cdot a = 0$.

If possible $\alpha \neq 0_F$ then $\exists\ \alpha^{-1} \in F$ s.t. $\alpha^{-1} \cdot \alpha = 1$

Then $\alpha^{-1} \cdot (\alpha \cdot a) = \alpha^{-1} \cdot 0 = 0$

ie. $(\alpha^{-1} \cdot \alpha) \cdot a = 0$

ie. $1 \cdot a = 0$

ie. $a = 0$

Hence either $\alpha = 0_F$ or $a = 0$.

(ii) Let $(V_4, o, \cdot)$ be a vector spance and $S_1 = \{a, e\}$ and $S_2 = \{b, e\}$ be two subspaces of $(V_4, o, \cdot)$

Then $S_1 \cup S_2 = \{e, a, b\}$ is not a subspace because $a\,o\,b = c \notin S_1 \cup S_2$.

(iii) The smallest subspace ~~generating~~ ~~by~~ containing the subset S is called subspace generating by S. It is denoted by $[S]$.

(iv). A set of vectors $S = \{a_1, a_2, \ldots a_n\}$ is called a linearly independent set if, for $\alpha_1, \alpha_2, \ldots \alpha_n \in F$.

$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n = 0$ will imply $\alpha_1 = \alpha_2 = \cdots = \alpha_n$.

(v). Let V be a vector space over a field F. The set of all linear transformations from V to F form a vector space. This vector space is called the dual vector space V^* of the vector space V.

(vi). If $S = \{x_1, x_2, \ldots, x_n\}$ be an orthonormal subset of an inner product space $(V, \langle\rangle)$, then for $\forall x \in V$, we have

$$|\langle x, x_1\rangle|^2 + |\langle x, x_2\rangle|^2 + \cdots + |\langle x, x_n\rangle|^2 \le \|x\|^2.$$

(vii). A subset S of an inner product space $(V, \langle\rangle)$ is said to be orthogonal set if distinct vectors in S are orthogonal i.e. $S = \{a_1, a_2, \ldots a_n\}$ is orthogonal if $\langle a_i, a_j\rangle = 0_F$, $i \ne j$.

A subset $S = \{a_1, a_2, \ldots a_n\}$ of $(V, \langle\rangle)$ is said to be orthonormal set if S is orthogonal set and norm of each vector in S is 1.

(viii). Let $T : V_1 \to V_2$ be a linear transformation, then

$$\text{Ker } T \overset{\text{def}}{=} \{x \in V_1 \mid T(x) = 0_2\}$$

2. (i) $\Rightarrow$ (ii)

Let $V = W_1 \oplus W_2$

then every element $x \in V$ can be written as $x = a_1 + a_2$ where a_1 & a_2 are unique element W_1 & W_2 respectively.

Then $V = W_1 + W_2$

Also let if possible $y (\ne 0) \in W_1 \cap W_2$

then $y \in W_1$ & $y \in W_2$

and then $y = \underset{\in W_1 + W_2}{y + 0}$ and $y = \underset{\in W_1 + W_2}{0 + y}$

This shows that non-zero element $y \in V$ has two different representation as one element of W_1 + one element of W_2, which contradicts that V is direct sum of W_1 & W_2.

Hence $y (\ne 0) \in W_1 \cap W_2$ is wrong

then $W_1 \cap W_2 = \{0\}$.

(ii) $\Rightarrow$ (i).

Let $V = W_1 + W_2$ & $W_1 \cap W_2 = \{0\}$.

Let if possible $a \in V$ ~~have~~ can be written as

$$a = a_1 + a_2 \quad \text{---}(*) \quad \text{and} \quad a = b_1 + b_2 \quad \text{---}(**)$$

where $a_1, b_1 \in W_1$ & $a_2, b_2 \in W_2$.

From $(*)$ and $(**)$

$$a_1 + a_2 = b_1 + b_2$$

ie. $a_1 - b_1 = a_2 - b_2 = x$ (say)

then $x = a_1 - b_1 \in W_1$ also $x = a_2 - b_2 \in W_2$

ie. $x \in W_1 \cap W_2$

But $W_1 \cap W_2 = \{0\}$, then $x = 0$

ie. $a_1 - b_1 = 0$ and $a_2 - b_2 = 0$

ie. $a_1 = b_1$ and $a_2 = b_2$

This shows that the two representations $(*)$ and $(**)$ for $a \in V$ are the same, ie. each $a \in V$ ~~to~~ can be uniquely written as one element of W_1 + one element of W_2.

Hence $V = W_1 \oplus W_2$.

3. Let $\dim W = l$

and $S = \{a_1, a_2, \cdots a_l\}$ be a basis of W.

Since S is L.I. subset of W, it is L.I. in V. So it can be extended to form a basis of V.

Let $S_1 = \{a_1, a_2, \cdots a_l, b_1, \cdots b_m\}$ be a basis of V.

then $\dim V = l + m$.

Consider $S_2 = \{W + b_1, W + b_2, \cdots, W + b_m\} \subseteq V/W$.

S_2 is L.I. subset of V/W.

Consider $\beta_1 (W + b_1) + \beta_2 (W + b_2) + \cdots + \beta_m (W + b_m) = W$ (zero of V/W)

then $W + (\beta_1 b_1 + \beta_2 b_2 + \cdots + \beta_m b_m) = W$

$$\beta_1 b_1 + \beta_2 b_2 + \cdots + \beta_m b_m \in W$$

$\beta_1 b_1 + \cdots + \beta_m b_m$ can be uniquely written as a linear combination of $a_1, a_2, \cdots a_l$.

ie $\beta_1 b_1 + \beta_2 b_2 + \cdots + \beta_m b_m = \alpha_1 a_1 + \cdots + \alpha_l a_l$

ie. $(-\alpha_1) a_1 + (-\alpha_2) a_2 + \cdots + (-\alpha_l a_l) + \beta_1 b_1 + \cdots + \beta_m b_m = 0$

then $-\alpha_1 = -\alpha_2 = \cdots = -\alpha_l = 0 = \beta_1 = \cdots = \beta_m$

(because $a_1, a_2, \cdots, a_l, b_1, \cdots b_m$ being basis elements are L.I.)

Hence S_2 is L.I.

$V/W = [S_2]$:

Let $W + x \in V/W$

then $x \in V$

$$x = \theta_1 a_1 + \cdots + \theta_l a_l + \phi_1 b_1 + \cdots + \phi_m b_m$$

$$\begin{aligned} W + x &= W + \theta_1 a_1 + \cdots + \theta_l a_l + \phi_1 b_1 + \cdots + \phi_m b_m \\ &= W + \phi_1 b_1 + \cdots + \phi_m b_m \quad \text{as } a_1, a_2, \cdots a_l \in W \\ &= \phi_1 (W + b_1) + \cdots + \phi_m (W + b_m) \\ &= \text{a lin. comb. of elements of } S_2. \end{aligned}$$

Thus S_2 is a basis of V/W.

Hence $\dim V/W = m = l + m - l$

$= \dim V - \dim W$.

4. $T_1 : V_1 \to V_2$ & $T_2 : V_2 \to V_3$

then $T_2 \circ T_1 : V_1 \to V_3$

Rank-Nullity threorem for T_1 and $T_2 \circ T_1$

$$\dim \operatorname{Ker} T_1 + \rho(T_1) = \dim V_1 \quad \text{--- (1)}$$

$$\dim \operatorname{Ker} T_2 \circ T_1 + \rho(T_2 \circ T_1) = \dim V_1 \quad \text{--- (2)}$$

from (1) & (2)

$$\dim \operatorname{Ker} T_2 \circ T_1 + \rho(T_2 \circ T_1) = \dim \operatorname{Ker} T_1 + \rho(T_1) \quad \text{--- (3)}$$

Now,

let $x \in \operatorname{Ker} T_1$

$T_1(x) = 0_2$

$T_2[T_1(x)] = T_2(0_2)$

$(T_2 \circ T_1)(x) = 0_3$

$x \in \operatorname{Ker} T_2 \circ T_1$

$\therefore \operatorname{Ker} T_1 \subseteq \operatorname{Ker} T_2 \circ T_1$

Since $\operatorname{Ker} T_1$ & $\operatorname{Ker} T_2 \circ T_1$ are subspaces of V_1, hence $\operatorname{Ker} T_1$ is a subspace of $\operatorname{Ker} T_2 \circ T_1$.

therefore $\dim \operatorname{Ker} T_1 \leq \dim \operatorname{Ker} T_2 \circ T_1 \quad$ --- (4)

From (3) & (4).

$$\rho(T_2 \circ T_1) \leq \rho(T_1).$$

Next; Since $\operatorname{Im} T_1 \subseteq V_2$

or $T_1(V_1) \subseteq V_2$

$T_2[T_1(V_1)] \subseteq T_2(V_2)$

$(T_2 \circ T_1)(V_1) \subseteq T_2(V_2)$

$\operatorname{Im} T_2 \circ T_1 \subseteq \operatorname{Im} T_2$

$\dim \operatorname{Im}(T_2 \circ T_1) \leq \dim \operatorname{Im} T_2$

ie. $\rho(T_2 \circ T_1) \leq \rho(T_2)$

Hence $\rho(T_2 \circ T_1) \leq \min(\rho(T), \rho(T_2))$.

5. Let $\alpha, \beta \in \mathbb{R}$, $(a_1, b_1), (a_2, b_2) \in \mathbb{R}^2$

$$T[\alpha(a_1,b_1) + \beta(a_2,b_2)] = T[(\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2)]$$

$$= (\alpha a_1 + \beta a_2 - \alpha b_1 - \beta b_2, \alpha b_1 + \beta b_2 - \alpha a_1 - \beta a_2, -\alpha a_1 - \beta a_2)$$

$$= (\alpha a_1 - \alpha b_1, \alpha b_1 - \alpha a_1, -\alpha a_1) + (\beta a_2 - \beta b_2, \beta b_2 - \beta a_2, -\beta a_2)$$

$$= \alpha(a_1 - b_1, b_1 - a_1, -a_1) + \beta(a_2 - b_2, b_2 - a_2, -a_2)$$

$$= \alpha T(a_1, b_1) + \beta T(a_2, b_2)$$

Therefore T is a linear transformation.

Let $(a, b) \in \operatorname{Ker} T$

$$T(a, b) = (0,0,0)$$

$$(a-b, b-a, -a) = (0,0,0)$$

$$\Rightarrow a = 0, \quad b = 0$$

$\therefore \operatorname{Ker} T = \{(0,0)\}$

nullity of $T = 0$.

By Rank-nullity theorem

$$\text{Rank } T = \dim \mathbb{R}^2 = 2.$$

Since $\mathbb{R}^2 = [\{(1,0), (0,1)\}$

$\therefore \operatorname{Im} T = [\{T(1,0), T(0,1)\}]$

ie Range $= [(1,-1,1), (-1,1,0)]$

6.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

The characteristic equation is

$$|A - \lambda I| = 0$$

ie. $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

ie. $\lambda = 1, 2, 3$

For $\lambda_1 = 1$; eigen vector is given by

$$(A - \lambda_1 I) X_1 = 0$$

ie. $\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

ie $-z_1 = 0$

& $x_1 + y_1 + z_1 = 0$

Hence $X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

For $\lambda_2 = 2$,

$$(A - \lambda_2 I) X_2 = 0$$

ie. $\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

ie $-x_2 - z_2 = 0$

$2x_2 + 2y_2 + z_2 = 0$

Hence $X_2 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

For $\lambda_3 = 3$,

$$(A - \lambda_3 I) X_3 = 0$$

ie. $$\begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

ie. $$-2x_3 - z_3 = 0$$

$$x_3 - y_3 + z_3 = 0$$

$$2x_3 + 2y_3 = 0$$

Hence $$X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

Then $$P = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix}$$

~~Hence~~ $\det P \neq 0$

Hence A is diagonalizable and

$$P^{-1} A P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D.$$

7. $$S = \{ \underbrace{(1,0,0)}_{=b_1}, \underbrace{(1,1,0)}_{=b_2}, \underbrace{(1,1,1)}_{=b_3} \}$$

(i) $$a_1 = \frac{b_1}{\|b_1\|} = \frac{(1,0,0)}{\sqrt{1^2+0+0}} = (1,0,0)$$

(ii) $$c_2 = b_2 - \langle b_2, a_1 \rangle a_1$$

$$= (1,1,0) - 1\,(1,0,0) = (0,1,0)$$

$$\therefore a_2 = \frac{c_2}{\|c_2\|} = \frac{(0,1,0)}{\sqrt{0+1+0}} = (0,1,0)$$

(iii) $c_3 = b_3 - \langle b_3, q_1 \rangle q_1 - \langle b_3, q_2 \rangle q_2$

$= (1,1,1) - 1(1,0,0) - 1(0,1,0)$

$= (0,0,1)$

$\therefore \; q_3 = \dfrac{c_3}{\|c_3\|} = \dfrac{(0,0,1)}{\sqrt{0+0+1}} = (0,0,1)$

Thus the orthonormal set is

$= \{(1,0,0), (0,1,0), (0,0,1)\}.$

8. <u>Cauchy - Schwarz's inequality</u>-

For any two vectors a and b of an inner product space $(V, \langle\,\rangle)$, we have

$$|\langle a, b \rangle| \leq \|a\| \cdot \|b\|$$

Proof. If $b = 0$, then the inequality satisfied in the form of equality. Therefore let us suppose $b \neq 0$

Consider $c = a - \lambda b$ where $\lambda = \dfrac{\langle a, b \rangle}{\|b\|^2}$

then $\|c\|^2 = \langle c, c \rangle$

$= \langle a - \lambda b, a - \lambda b \rangle$

$= \langle a, a \rangle - \bar{\lambda} \langle a, b \rangle - \lambda \langle b, a \rangle + \lambda \bar{\lambda} \langle b, b \rangle$

$= \|a\|^2 - \dfrac{\overline{\langle a, b \rangle}}{\|b\|^2} \langle a, b \rangle - \dfrac{\langle a, b \rangle}{\|b\|^2} \langle b, a \rangle + \dfrac{|\langle a, b \rangle|^2}{\|b\|^4} \|b\|^2$

$= \|a\|^2 - \dfrac{|\langle a, b \rangle|^2}{\|b\|^2}$

Since $\|c\|^2 \geq 0_F$

$$\|a\|^2 - \frac{|\langle a, b\rangle|^2}{\|b\|^2} \geq 0_F$$

ie. $$\|a\|^2 \|b\|^2 - |\langle a, b\rangle|^2 \geq 0_F$$

ie. $$|\langle a, b\rangle| \leq \|a\| \cdot \|b\|$$